Doing the Twist: Diagonal Meshes are Isomorphic to Twisted Toroidal Meshes

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Abstract—We show that a $k \times n$ diagonal mesh is isomorphic to a $\frac{n+k}{2} \times \frac{n+k}{2} - \frac{n-k}{2} \times \frac{n-k}{2}$ twisted toroidal mesh, *i.e.* a network similar to a standard $\frac{n+k}{2} \times \frac{n+k}{2}$ toroidal mesh, but with opposite handed twists of $\frac{n-k}{2}$ in the two directions, which results in a loss of $(\frac{n-k}{2})^2$ nodes.

Index Terms—interconnection networks, grid networks, mesh-connected topologies, diagonal mesh, toroidal mesh.

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Tang and Padubidri [1] analyze the diagonal mesh suggested by Arden, finding non-square diagonal meshes superior to the usual toroidal mesh in a number of respects. In figure 1a, the 5×5 diagonal mesh from their first figure (which is square, and thus not covered by their claims) is drawn. As shown by figure 1b, this network is isomorphic[1] to a standard 5×5 toroidal mesh. A non-square diagonal mesh is not in general isomorphic to a standard toroidal mesh, but instead to a *twisted* toroidal mesh, a class of network pictured in figure 2.

As proven diagramatically in figure 3, any $k \times n$ diagonal mesh (n and k are necessarily odd, and without loss of generality $k \leq n$) is isomorphic to a $\frac{n+k}{2} \times \frac{n+k}{2} - \frac{n-k}{2} \times \frac{n-k}{2}$ twisted toroidal mesh. This twisted toroidal mesh is like a standard $\frac{n+k}{2} \times \frac{n+k}{2}$ toroidal mesh, except that the edges are joined with twists of opposite handedness of $\frac{n-k}{2}$ in the two directions, and there is a consequent loss of an $\frac{n-k}{2} \times \frac{n-k}{2}$ corner, as shown in figure 2. A convenient notation for a $k \times n$ toroidal mesh with twists of a and b in the two directions is $k \times n \pm a \times b$, with $+$ if the twists have the same handedness and $-$ if they have opposite handedness. This notation serves a dual purpose, as such networks have $kn \pm ab$ nodes.

This isomorphism simplifies the analysis of diagonal meshes. For instance, for a large network, holding the number of nodes in a $k \times n \pm a \times b$ twisted toroidal mesh fixed while allowing k, n, a, and b to vary, it is elementary to see that the bisection width and diameter reach extremes at the discontinuities of the domain, namely configurations of the form $n \times n \pm \frac{n}{2} \times \frac{n}{2}$. The extreme which optimizes performance is $n \times n - \frac{n}{2} \times \frac{n}{2}$, and is isomorphic to a $k \times 3k$ diagonal mesh, and therefore to a $2k \times 2k - k \times k = k \times 3k - 0 \times 2k = k \times 3k + 0 \times k$ twisted toroidal mesh.

We call a network *singularly transversible* when, by moving repeatedly in one direction, all nodes will be visited. This property can be useful for testing, power distribution, diagnosis, and initialization. A $k \times n \pm a \times b$ twisted toroidal mesh is singularly transversible exactly when $\gcd(k, a) = \gcd(n, b) = 1$. This implies that a $k \times n$ diagonal mesh is singularly transversible when n and k are relatively prime.

Consider a $k \times n - a \times b$ twisted toroidal mesh as an Abelian group. This group can be generated by the two elements N and E. Two identities suffice to characterize its properties: $E^n = N^a$ and $N^k = E^b$ and it is there-

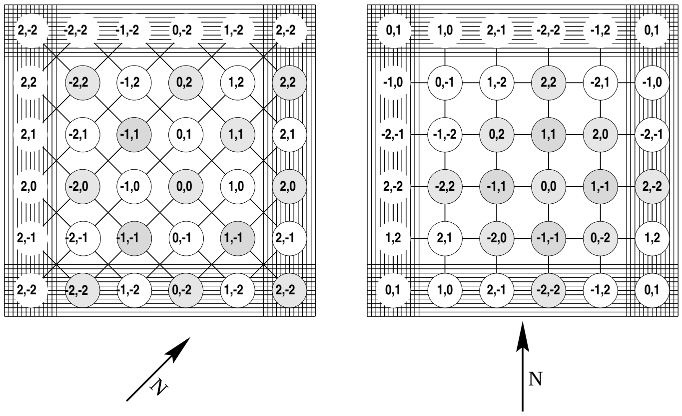

Fig. 1. (a) 5×5 diagonal mesh. The edge behavior is shown by ghost units and corresponding regions. (b) 5×5 toroidal mesh. As demonstrated by the node labels, these two networks are isomorphic.

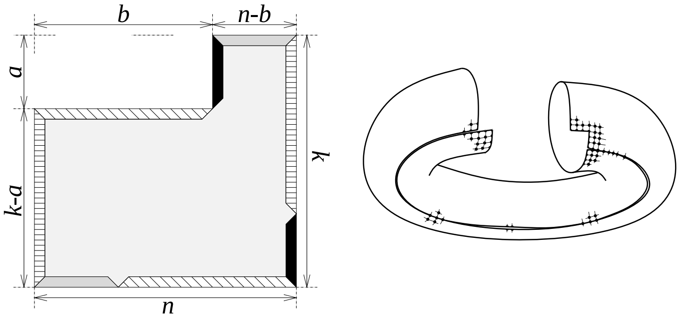

Fig. 2. Removing an $a \times b$ rectangular area from the corner of a $k \times n$ rectangular mesh before joining the edges to form a torus allows opposite handed twists of a and b nodes to be made in the two directions, resulting in an $k \times n - a \times b$ twisted toroidal mesh, which has $kn - ab$ nodes. Edge identifications are shown on the left using shaded regions, while the process of rolling the sheet up is shown on the right.

[1] Isomorphic in the sense that there exists a bijective mapping of nodes to nodes, edges to edges, and directions to directions that preserves all mathematical properties.

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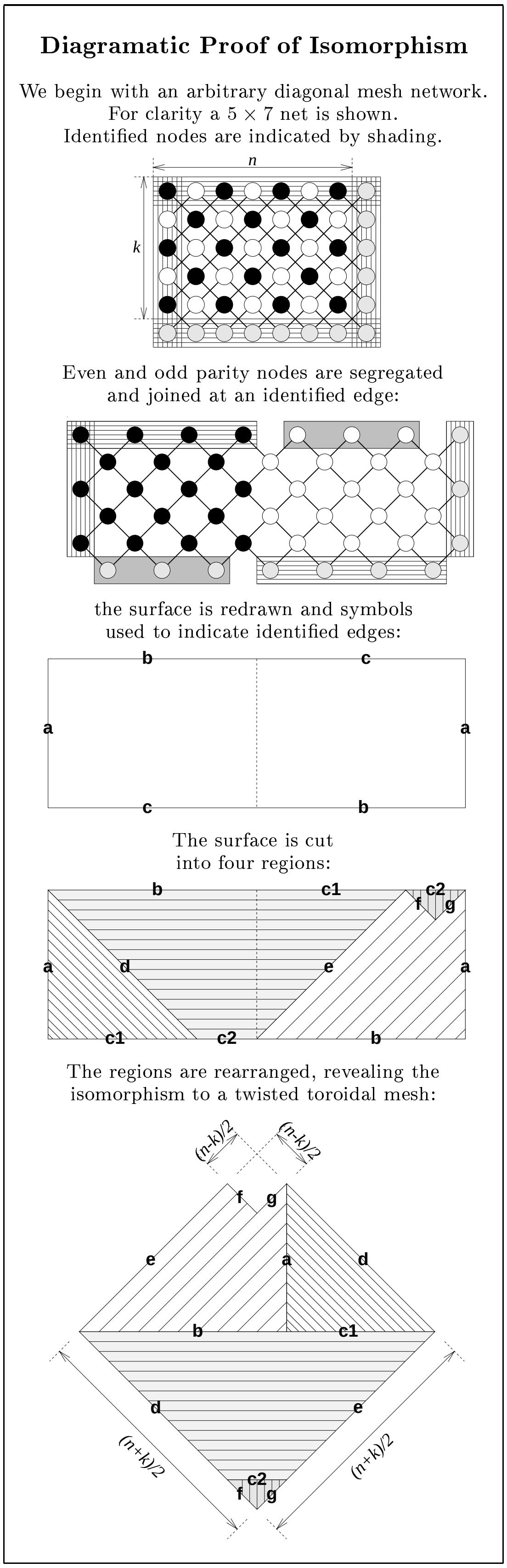

Fig. 3. This diagram sketches a simple proof that a $k \times n$ diagonal mesh network is isomorphic to a $\frac{n+k}{2} \times \frac{n+k}{2} - \frac{n-k}{2} \times \frac{n-k}{2}$ twisted toroidal mesh.

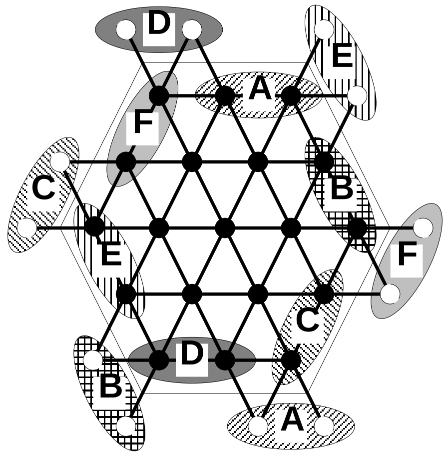

Fig. 4. Hex-mesh regular hexagonal arrays can be rolled into twisted toruses, as in this 19-element E3 network [2].

fore isomorphic to the quotient group $\mathcal{Z}^2/A$ where A is the subgroup of $\mathcal{Z}^2$ generated by (n,a) and (k,b). We would like to find a canonical (up to rotation) representation for this twisted toroidal mesh. Such a representation is $k' \times n' - 0 \times b'$ where

$$k' = \gcd(k,a) \qquad\qquad n' = \frac{kn - ab}{k'}$$

and to preserve the group identities, it is necessary that $b' \frac{k}{k'} = b \pmod{n'}$ and $b' \frac{a}{k'} = n \pmod{n'}$. Using the Chinese remainder theorem we can find integers x and y such that $x\frac{k}{k'} + y\frac{a}{k'} = 1$, so

$$b' = xb + yn \pmod{n'}$$

This gives a simple algorithm for testing twisted toroidal mesh isomorphism.

A twisted toroidal topology was used as the routing network of the FAIM-1 parallel computer [2]. Figure 4 shows the 19-element E3 hex-mesh toroidal network they built. If all the links in one of the three directions are removed, what remains is a $5 \times 5 - 2 \times 3$ twisted toroidal mesh.

References

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